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The volume of hyperbolic Coxeter polytopes of even dimension

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1. INTRODUCTION

Let H^n denote hyperbolic space of dimension n, and let S be an index set for a finite collection of open half spaces H_s^+ in H^n bounded by codimension one hyperplanes H_s. We assume that for all distinct $s, t \in S$ either $H_s \cap H_t$ is not empty and the (interior) dihedral angle of $H_s^+ \cap H_t^+$ along $H_s \cap H_t$ has size π/m_{st} for certain integers $m_{st} = m_{ts} \geq 2$, or $H_s \cap H_t$ is empty while $H_s^+ \cap H_t^+$ is not empty. In the latter case we put $m_{st} = m_{ts} = \infty$ and we also put $m_{ss} = 1$. Under these assumptions the intersection $C = \bigcap_s H_s^+$ is not empty, and its closure D is called a hyperbolic Coxeter polytope.

By abuse of notation let $s \in S$ also denote the reflection of H^n in the hyperplane H_s. Now the group W of motions of H^n generated by the reflections $s \in S$ is discrete, and D is a strict fundamental domain for the action of W on H^n. Moreover (W, S) is a Coxeter group with Coxeter matrix $M = (m_{st})$, i.e. W has a presentation with generators $s \in S$ and relations $(st)^{m_{s,t}} = 1$ for $s, t \in S$. Let $\ell(w)$ denote the length of $w \in W$ with respect to the generating set S, and let $P_W(t) \in \mathbb{Z}[[t]]$ be the Poincaré series of W defined by $P_W(t) = \sum_w t^{\ell(w)}$.

Theorem. *If D has finite hyperbolic volume then we have the relation*

$$\frac{1}{P_W(1)} = \begin{cases} \dfrac{(-1)^{n/2} 2 \operatorname{vol}_n(D)}{\operatorname{vol}_n(S^n)} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

For D compact this can be derived from the work by Serre on the cohomology of discrete groups [Se]. Here we obtain the result as a consequence of the differential volume formula of Schläfli. This method was inspired by a recent paper of Kellerhals where $\mathrm{vol}_{2n}(D)$ was computed in case D is a (possibly simply or doubly truncated) orthoscheme [Kel, IH].

The above theorem is essentially just a specialization of the Gauss–Bonnet theorem to the present situation [Ho, Fe, AW, Ch, Sa]. Nevertheless I have found it worthwhile to write these things up in some detail in order to emphasize the elementary nature of this approach. For partial results on the computation or $\mathrm{vol}_n(D)$ for n odd one is referred to [Ke2, Ke3] and the references mentioned there.

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2. THE DIFFERENTIAL VOLUME FORMULA OF SCHLÄFLI AND SOME CONSEQUENCES

Let D be a spherical or a hyperbolic simplex of dimension n. The codimension one faces of D are labeled D_s for $s \in S$ an index set of cardinality $n+1$. The faces of D are of the form $D_J = \bigcap_{s \in J} D_s$ with J a proper subset of S. Clearly D_J has codimension $|J|$. The interior angle of D along D_J is denoted by D^J. Clearly D^J is a simplicial cone in a euclidean space of dimension $|J|$, and it also determines a spherical simplex $D^J \cap S^{|J|-1}$ of dimension $|J|-1$. Note that the simplex D is determined up to motions by its dihedral angles $\alpha_J := \mathrm{vol}_1(D^J \cap S^1)$ with $J \subset S$ and $|J| = 2$.

Theorem (differential volume formula of Schläfli): *For $J \subset S$ with $|J| = 2$ we have*

$$\frac{\partial}{\partial \alpha_J}(\mathrm{vol}_n(D)) = \frac{\varepsilon}{n-1}\mathrm{vol}_{n-2}(D_J)$$

where $\varepsilon = 1$ if D is a spherical simplex and $\varepsilon = -1$ if D is a hyperbolic simplex.

In the spherical case this formula was found by Schläfli in 1852 [Sc]. The three dimensional hyperbolic version goes back to Lobachevsky [Co]. A nice and simple proof of this formula (valid in both spherical and hyperbolic case) was given by Kneser [Kn, BH].

Corollary. *Renormalize $\mathrm{vol}_n(D)$ by putting $G_n(D) = \mathrm{vol}_n(D)/\mathrm{vol}_n(S^n)$. For $J \subset S$ with $|J| = 2$ we have*

$$\frac{\partial G_n(D)}{\partial G_1(D^J \cap S^1)} = \varepsilon G_{n-2}(D_J).$$

Proof. This is just a reformulation of the differential volume formula using that $\mathrm{vol}_n(S^n) = 2\pi^{(n+1)/2}\Gamma((n+1)/2)^{-1}$. $\square$

Theorem (reduction formula): *With the convention $G_{-1}(\cdot) = 1$ we have*

$$\varepsilon^{n/2}(1+(-1)^n)G_n(D) = \sum_{I \subsetneq S} (-1)^{|I|} G_{|I|-1}(D^I \cap S^{|I|-1}).$$

Proof. By induction on the dimension n of D. The case $n = 1$ is trivial. In case $n = 2$ and D is a triangle with angles α, β, γ the equality of the left hand side $(2\varepsilon/4\pi)\mathrm{vol}_2(D)$ and the right hand side $(1 - \frac{3}{2} + (1/(2\pi))(\alpha+\beta+\gamma))$ is a familiar formula. Now suppose $n \geq 3$. Suppose $J \subset S$ with $|J| = 2$. We will check that the derivatives of both sides with respect to the renormalized dihedral angle $G_1(D^J \cap S^1)$ of D along D_J are equal. This implies that the formula is correct upto an additive constant. Indeed for the left hand side we get

$$\varepsilon^{n/2}(1+(-1)^n)\frac{\partial G_n(D)}{\partial G_1(D^J \cap S^1)} = \varepsilon^{(n-2)/2}(1+(-1)^{n-2})G_{n-2}(D_J),$$

and for the right hand side we get

$$\begin{aligned}&\sum_{J \subset I \subsetneq S} (-1)^{|I|} \frac{\partial G_{|I|-1}(D^I \cap S^{|I|-1})}{\partial G_1(D^J \cap S^1)} \\ &= \sum_{K \subsetneq S \setminus J} (-1)^{|K|} G_{|K|-1}((D_J)^K \cap S^{|K|-1}).\end{aligned}$$

Here we have used that for $J \subset I \subsetneq S$ we have $(D^I)_J = (D_J)^{I \setminus J}$. Hence we arrive at the reduction formula for the face D_J. It remains to check the constant. In the spherical case we take D a simplex with all dihedral angles equal to $\pi/2$. Hence $G_n(D) = 2^{-n-1}$ and the reduction formula reduces in this case to the correct identity $(1+(-1)^n)2^{-n-1} = \sum_{k=0}^{n} \binom{n+1}{k}(-\frac{1}{2})^k$. This proves the reduction formula for D a spherical simplex. Taking a shrinking sequence of spherical simplices it follows that the angle sum on the right hand side of the reduction formula vanishes for a euclidean simplex D. In turn this also shows that the constant matches for D a hyperbolic simplex. $\square$

For spherical simplices the reduction formula is due to Schläfli. Unaware of Schläfli's work the reduction formula was rediscovered by Poincaré with a different and elegant proof [Po]. The extension from a spherical to a hyperbolic simplex was made by Hopf [Ho].

Corollary. *Suppose D is a convex hyperbolic polytope with finite volume and of dimension n. Denote by $F(D)$ the collection of faces of D, and for F a face of D of codimension $|F|$ write D^F for the interior angle (in $\mathbb{R}^{|F|}$) of D along F. Then the following reduction formula holds*

$$2\cos\left(\frac{n\pi}{2}\right)G_n(D) = \sum_{F \in F(D)} (-1)^{|F|} G_{|F|-1}(D^F \cap S^{|F|-1}).$$

Proof. If D is unbounded but with finite volume then some vertices of D lie on the boundary of H^n. At such a cusp like vertex the size of the interior angle of D equals zero. Hence by continuity we may assume that D is bounded. For $D = \bigcup D_i$ a simplicial subdivision of D we get

$$\begin{aligned} 2\cos\left(\frac{n\pi}{2}\right)G_n(D) &= \sum_i 2\cos\left(\frac{n\pi}{2}\right)G_n(D_i) \\ &= \sum_i \sum_{I \subsetneq S_i} (-1)^{|I|}G_{|I|-1}(D_i^I \cap S^{|I|-1}) \\ &= \sum_F \sum_{(i,I)\sim F} (-1)^{|I|}G_{|I|-1}(D_i^I \cap S^{|I|-1}) \end{aligned}$$

where F runs over the faces of D, and we write $(i,I) \sim F$ if the relative interior of $D_{i,I}$ is contained in the relative interior of F. Since for fixed $(i,I) \sim F$ the interior angles D_j^J with $D_{j,J} = D_{i,I}$ make up an interior angle $D^F \times \mathbb{R}^{|I|-|F|}$ we conclude that

$$\sum_{(i,I)\sim F} (-1)^{|I|}G_{|I|-1}(D_i^I \cap S^{|I|-1}) = (-1)^{|F|}G_{|F|-1}(D^F \cap S^{|F|-1}),$$

because the Euler characteristic of the relative interior of F is equal to $(-1)^{\dim(F)}$. $\square$

A direct consequence of this corollary is the Gauss–Bonnet formula for hyperbolic space forms originally derived by Hopf along these lines.

Corollary. *For Γ a group acting discretely on H^{2n} with a smooth compact oriented quotient $\Gamma \backslash H^{2n}$ the Euler characteristic $\chi(\Gamma \backslash H^{2n})$ of $\Gamma \backslash H^{2n}$ is given by*

$$\chi(\Gamma \backslash H^{2n})\mathrm{vol}_{2n}(S^{2n}) = (-1)^n 2\mathrm{vol}_{2n}(\Gamma \backslash H^{2n}).$$

3. HYPERBOLIC COXETER GROUPS

Let $M = (m_{st})$ be a Coxeter matrix, i.e. $m_{ss} = 1$ for all $s \in S$ and $m_{st} = m_{ts} \in \{2, 3, \dots, \infty\}$ for all $s, t \in S$. Let $G = (g_{st})$ with $g_{st} = -2\cos(\pi/m_{st})$ if m_{st} is finite, and if $m_{st} = \infty$ let $g_{st} = -2c_{st}$ with $c_{st} = c_{ts} \geq 1$ an additional parameter. Let V be a real vector space with basis $\{\alpha_s;\ s \in S\}$, and equip V with a symmetric bilinear form by $(\alpha_s, \alpha_t) = g_{st}$. For $\alpha \in V$ with $(\alpha, \alpha) = 2$ let $r_\alpha \in GL(V)$ be the orthogonal reflection in the hyperplane perpendicular to α: $r_\alpha(\lambda) = \lambda - (\lambda, \alpha)\alpha$ for $\lambda \in V$. Let (W, S) be the Coxeter system corresponding to the matrix M. The homomorphism $\sigma : W \to GL(V)$ defined by $\sigma(s) = r_s$ for $s \in S$ (r_s is short for r_{α_s}) is the (possibly deformed) geometric representation. The theory as developed for example in [Hu, Chapter 5] for the ordinary (i.e. $c_{st} = 1$ if $m_{st} = \infty$) geometric representation goes through verbatim in the present situation.

Let V^* be the dual vector space of V and $\{\xi_s;\ s \in S\}$ the basis of V^* dual to $\{\alpha_s;\ s \in S\}$. Hence $(\xi_s, \alpha_t) = \delta_{st}$ for all $s, t \in S$ where $(\cdot, \cdot)$ also denotes the pairing between V^* and V. For $J \subset S$ we put

$$C_J := \left\{ \sum_s x_s \xi_s;\ x_s = 0 \text{ if } s \in J,\ x_s > 0 \text{ if } s \notin J \right\}.$$

Clearly $C_S = \{0\}$ and $C := C_\emptyset$ is an open simplicial cone. The closure D of C admits a partition $D = \bigcup_J C_J$ and C_J is a face of D of codimension $|J|$. For $w \in W$ and $\xi \in V^*$ write $w(\xi)$ for $\sigma^*(w)(\xi)$. The Tits cone

$$U := \bigcup_w w(D)$$

is a convex cone in V^*. Moreover $C_I \cap w(C_J)$ is not empty for $I, J \subset S$ and $w \in W$ if and only if $I = J$ and $w \in W_J$. Here W_J is the (parabolic) subgroup of W generated by J.

Let V' be the orthocomplement in V^* of the kernel K of the symmetric bilinear form $(\cdot, \cdot)$ on V. Clearly V/K inherits a canonical non-degenerate symmetric bilinear form from V, and since V/K and V' are dual vector spaces this form can be transferred to V'. By abuse of notation we denote this form again by $(\cdot, \cdot)$. For $J \subset S$ let G_J denote the submatrix of G with indices taken from J.

Proposition. *Suppose the matrix G is indecomposable and has smallest eigenvalue < 0. Let $J \subset S$ such that G_J is positive definite. Then there exists a vector $\xi_J \in C_J \cap V'$ with $(\xi_J, \xi_J) < 0$, and $C_J \cap V'$ is a face of the polyhedral cone $D \cap V'$ of codimension $|J|$.*

Proof. Let $J \subset S$ such that G_J is positive definite. Let 1_J denote the matrix with 1 on the places ss for $s \notin J$ and 0 elsewhere. For $t \in \mathbb{R}$ sufficiently large the matrix $G + t1_J$ is positive definite, and let t_J be the infimum of these t. Clearly $t_J > 0$ and the matrix $G + t_J 1_J$ is positive semidefinite with nonzero kernel. By the Perron–Frobenius lemma [Hu, Section 2.6] the kernel is one dimensional and spanned by a vector x_J with all coordinates $x_{J,s} > 0$ for $s \in S$. Now put

$$\alpha_J := \sum_{s \in S} x_{J,s} \alpha_s \in V, \qquad \xi_J := \sum_{s \notin J} x_{J,s} \xi_s \in V^*.$$

Then we have on the one hand (the brackets denote the bilinear form on V)

$$(\alpha_J, \alpha_s) = 0 \quad \text{for } s \in J$$
$$(\alpha_J, \alpha_s) = -t_J x_{J,s} \quad \text{for } s \notin J,$$

and on the other hand (the brackets denote the pairing between V^* and V)

$$(\xi_J, \alpha_s) = 0 \quad \text{for } s \in J$$
$$(\xi_J, \alpha_s) = x_{J,s} \quad \text{for } s \notin J.$$

Hence $(\alpha_J, \alpha) + (t_J \xi_J, \alpha) = 0$ for all $\alpha \in V$. In turn this implies $\xi_J \in V'$ and $(\xi_J, \xi_J) = -t_J^{-1}(\alpha_J, \xi_J) = -t_J^{-1} \sum_{s \notin J} x_{J,s}^2 < 0$. Finally the codimension of C_J as face of D and the codimension of $C_J \cap V'$ as face of $D \cap V'$ is equal, because the intersection $C_J \cap V'$ is transversal (immediate by induction on $|J|$). $\square$

Remark. Suppose the matrix G is indecomposable and has smallest eigenvalue < 0. If $J \subset S$ such that G_J is positive semidefinite then it may happen that

$C_J \cap V'$ is empty. However it can be shown that there exist a proper subset I of S containing J and a vector $\xi_I \in C_I \cap V'$ with $(\xi_I, \xi_I) \leq 0$.

Definition. The matrix G is called hyperbolic if G is indecomposable, and the smallest eigenvalue of G is < 0, and all remaining eigenvalues of G are ≥ 0. The (irreducible) Coxeter group (W, S) with Coxeter matrix M is called hyperbolic if there exists a hyperbolic matrix G compatible with M.

From now on assume that the matrix G is hyperbolic. The set $\{\xi \in V';\ (\xi, \xi) < 0\}$ consists of two connected components, and the one containing the point $\xi_\emptyset$ is denoted by V'_+.

Proposition. *The open cone V'_+ is contained in $U \cap V'$.*

Proof. Let $R = \{w(\alpha_s);\ w \in W,\ s \in S\}$ be the (normalized) root system in V, and let $R' \subset V'$ be the 'restriction' of R to V'. It is not hard to show (and for this G need not be hyperbolic) that R' is a discrete subset of $\{\xi \in V';\ (\xi, \xi) = 2\}$. In turn this implies that the reflection hyperplanes $H_\alpha = \{\xi \in V';\ (\xi, \alpha) = 0\}$ for $\alpha \in R$ are locally finite on V'_+. Now for $\xi \in V'$ we have the familiar criterium: $\xi \in U$ if and only if the segment $[\xi_\emptyset, \xi]$ intersects only finitely many reflection hyperplanes H_α for $\alpha \in R$. Hence V'_+ is contained in $U \cap V'$. □

Theorem. *The intersection $C_J \cap V'_+$ is not empty if and only if the matrix G_J is positive definite, and in that case $C_J \cap V'_+$ is a face of $D \cap V'_-$ of codimension $|J|$.*

Proof. The stabilizer of $\xi \in V'_+$ in the Lorentz group $O(V') = \{g \in GL(V');\ g$ preserves $(\cdot, \cdot)\}$ is compact, and hence the stabilizer of $\xi \in V'_+$ in W is finite (as the intersection of a compact with a discrete set). Hence if $C_J \cap V'_+$ is not empty then W_J is finite, which is equivalent with G_J being positive definite. The converse and the remaining part of the theorem follows from the first proposition of this section. □

Now let $H = \{\xi \in V'_+;\ (\xi, \xi) = -1\}$ be hyperbolic space. The hyperbolic Coxeter polytope $D \cap H$ is a fundamental domain for the action of the group W on H. Moreover each action of an irreducible reflection group on hyperbolic space arises in this way.

Conclusion. The Coxeter polytope $D \cap H$ is compact if and only if $C_J \cap V'$ is empty for all $J \subsetneq S$ with G_J not positive definite. Also $D \cap H$ has finite hyperbolic volume if and only if $C_J \cap V'$ is empty for all $J \subsetneq S$ with G_J indefinite.

In some examples it can be cumbersome to check the above conditions. The results of this section are essentially due to Vinberg, and we refer to the nice survey paper [Vi] for a discussion of examples.

4. PROOF OF THE THEOREM

Let (W, S) be an arbitrary Coxeter group, and write $P_W(t) = \sum_w t^{\ell(w)}$ for the Poincaré series of (W, S). The following formula due to Steinberg [St] gives an effective way of computing $P_W(t)$ by induction on $|S|$.

Proposition. *The Poincaré series $P_W(t)$ is a rational function of t satisfying*

$$\frac{1}{P_W(t^{-1})} = \sum_{J \subset S,\, W_J \text{ finite}} (-1)^{|J|} \frac{1}{P_{W_J}(t)}.$$

Proof. For $X \subset W$ write $P_X(t) = \sum_{w \in X} t^{\ell(w)}$. If for $J \subset S$ we write $W^J := \{w \in W;\ \ell(ws) > \ell(w)\ \forall s \in J\}$ for the minimal length representatives for the left cosets of W_J then $P_W(t) = P_{W_J}(t) P_{W^J}(t)$. For $J \subset S$ with W_J finite let $N(J)$ be the length of the longest element $w_0(J)$ in W_J. If $J(w) := \{s \in S;\ \ell(ws) < \ell(w)\}$ for $w \in W$ then $w \in W^J w_0(J)$ for some $J \subset S$ with W_J finite precisely when $J \subset J(w)$. We claim that

$$\sum_{J \subset S,\, W_J \text{ finite}} (-1)^{|J|} P_{W^J w_0(J)}(t) = 1.$$

Indeed the contribution of $w \in W$ to the sum on the left hand side equals $\sum_{J \subset J(w)} (-1)^{|J|}$, which equals 0 unless $J(w)$ is empty. But $J(w)$ is empty precisely when $w = 1$ and the contribution becomes 1. Now we have

$$P_{W^J w_0(J)}(t) = t^{N(J)} P_{W^J}(t) = t^{N(J)} \frac{P_W(t)}{P_{W_J}(t)} = \frac{P_W(t)}{P_{W_J}(t^{-1})},$$

and the desired formula

$$\sum_{J \subset S,\, W_J \text{ finite}} (-1)^{|J|} \frac{1}{P_{W_J}(t^{-1})} = \frac{1}{P_W(t)}$$

follows. □

The theorem of the introduction follows by applying the reduction formula of Section 2 to the Coxeter polytope with finite hyperbolic volume. Combining the theorem of Vinberg of Section 3 with the above formula of Steinberg (evaluated at $t = 1$) indeed proves the desired formula.

5. FINAL REMARKS

Suppose G is a discrete cocompact group of isometries of hyperbolic space H^n. Fix a generic point $x \in H^n$ with trivial stabilizer in G, and put

$$D = \{y \in H^n;\ d(y, x) \leq d(y, gx)\ \forall g \in G\}$$

with d the hyperbolic distance. The compact convex polytope D is a fundamental domain for the action of G on H^n, and the set

$$S = \{g \in G;\ g(D) \cap D \text{ has codimension one}\}$$

is a finite set of generators for G. Let $\ell = \ell_S$ denote the length function on G with respect to S. It was shown by Cannon that the growth series

$$P_{G,S}(t) = \sum_{g \in G} t^{\ell(g)}$$

is the power series around $t = 0$ of a rational function in t [Ca]. Now it is a natural question whether the theorem from the introduction remains valid in the present situation. Although this seems to be quite often the case, there are counterexamples for dimension $n = 2$ [Pa, FP]. We refer to the latter paper for a further discussion of this problem.

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